Proceedings of the Jangjeon Mathematical Society
27 (2024), No. 2, pp. 135 - 144
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2024.27.2.135

ON S-ENDO-NOETHERIAN RINGS

MOHAMMED ISSOUAL AND SALAH EDDINE MAHDOU

Abstract. A commutative ring A is said to be an S-endo-Noetherian ring, where S is a multiplicative subset of A, if the chain of annihilators $ann_A(a_1) \subseteq ann_A(a_2) \subseteq \cdots$ is S-stationary; (i.e, there exist $s \in S$ and a positive integer n such that $s.(ann_A(a_k)) \subseteq ann_A(a_n)$ for each $k \geq n$) for each sequence $(a_k)_{k\geq 1}$ of elements of A. We study several properties of an S-endo-Noetherian ring. Among other results, the transfer of this property to the amalgamated duplication of A along an ideal I ($A \bowtie I$) is investigated, as well as the necessary and sufficient conditions for $R[[X]]$ (resp., $R[X]$) to be an S-endo-Noetherian ring.

Mathematics Subject Classification (2020): 13A15; 13F20; 13F25; 13E99.
Keywords: endo-Noetherian ring, S-endo-Noetherian ring, S-Noetherian ring, amalgamated duplication along an ideal.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with identity and all modules are nonzero unitary. Our purpose is to generalize the study of endo-Noetherian rings to the class of S-endo-Noetherian rings. The notion of an endo-Noetherian rings was introduced by B. Gouaid, A. Hamed and A. Benhissi [12].

Let A be a ring and M be an A-module. We say that M is an endo-Noetherian module if it satisfies the ascending chain condition for the kernels of the endomorphism; i.e, each increasing sequence of the form $\ker(f_1) \subseteq \ker(f_2) \subseteq ....$ is stationary, where $(f_k)_{k\geq 1}$ is a sequence of endomorphisms of M. A ring A is said to be endo-Noetherian if it is endo-Noetherian as a A-module; i.e, each chain of annihilators $ann(a_1) \subseteq ann(a_2) \subseteq \cdots$ is stationary. The class of S-endo-Noetherian rings includes the class of S-Noetherian rings [12].

Let A be a ring and E be an A-module. The following ring construction, called the trivial ring extension of A by E (also called the idealization of E), was introduced by Nagata [19, page 2]. It is the ring $A(+)E$ whose underlying abelian group is $A \times E$ with multiplication given by $(a, e)(b, f) = (ab, af + be)$. Trivial ring extensions have been studied extensively; and considerable work has been concerned with these extensions. Mainly, trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory.

Let A be a ring and I an ideal of A. The following ring construction, called the amalgamated duplication of A along I, was introduced by D'Anna in [8]. It is the subring $A \bowtie I$ of $A \times A$ given by

$$A \bowtie I := \{(a, a+i) \mid a \in A \text{ and } i \in I\}$$

This extension has been studied, in the general case and from the different point of view of pullbacks, by D'Anna and Fontana [10]. Also see the survey article [11] for more details on general constructions. If $I^2 = 0$, then $A \bowtie I$ coincides with Nagata's idealization ($A \propto E$). One main difference of this construction, with respect to the idealization, is that the ring $A \bowtie I$ can be reduced (and it is always reduced if A is an integral domain). Our paper consists of three sections.

In section 2, we study several properties of S-endo-Noetherian rings. Among other thing, we show that, if a ring A satisfies the property $S-(*)$, then it satisfies the $S-accr$ condition (we say that A satisfies the $S-accr$ condition if the ascending chain of colon ideals of the form $(I:a) \subseteq (I:a^2) \subseteq (I:a^3) \subseteq \cdots$ is S-stationary for each $a \in A$ and every ideal I of A), where the $S-(*)$ property means: every increasing sequence of the form $(I:a_1) \subseteq (I:a_2) \subseteq \cdots$ is S-stationary, where I is an ideal of A and $(a_k)_{k\geq 1}$ a sequence of elements of A. We also show that A satisfies the property $S-(*)$ if and only if A/I is an $\overline{S}$-endo-Noetherian ring for each ideal I of A. Then we provide an answer to the question of when the amalgamated duplication, of A along an ideal $I, A \bowtie I$ is an $S \times S$-endo-Noetherian ring. It is well known that a regular element of A is by definition a not a zero-divisor. Also, we say that an ideal I is a regular ideal if it contains a regular element of A. We prove that if I is a regular ideal of A, then A is an S-endo-Noetherian ring if and only if $A \bowtie I$ is an $S \times S$-endo-Noetherian ring.

In section 3, we find a necessary and sufficient conditions for a formal power series ring $R[[X]]$ and a a polynomial ring $R[X]$ to be an S-endo-Noetherian ring.

2. General properties

Recall from [20] that a ring A is said to be *endo-Noetherian* if the increasing sequence $ann_A(a_1) \subseteq ann_A(a_2) \subseteq ....$ is stationary for any sequence $(a_k)_k$ of elements of A. Let S be a multiplicative subset of A. According to [13], an increasing sequence $(I_k)_k$ of ideals of A is called S-stationary if there exist a positive integer n and $s \in S$ such that for each $k \geq n, sI_k \subseteq I_n$. We start this section by introducing the following definition in order to generalize several well-known results about endo-Noetherian rings.

Definition 2.1. Let A be a commutative ring and S a multiplicative subset of A. We say that A is an S-endo-Noetherian ring if the chain of annihilators $ann_A(a_1) \subseteq ann_A(a_2) \subseteq \cdots$ is S-stationary for each sequence $(a_k)_k$ of elements of A.

In the following examples we illustrate the link between endo-Noetherian rings, S-Noetherian rings and S-endo-Noetherian rings. As usual, A denotes a commutative ring and S be a multiplicative subset of A,

Example 2.2. If A is endo-Noetherian, then A is an S-endo-Noetherian ring.

Example 2.3. If S consists of units of A, then A is an S-endo-Noetherian ring if and only if A is an endo-Noetherian ring.

Example 2.4. If A is an S-Notherian ring, then A is an S-endo-Notherian ring. Indeed, by [13, Remark 2.3], every increasing sequence of ideals of A is S-stationary. But the converse is false. Let K be a field and E be a K-vector space of infinite dimension. Set $K(+)E$ to be the trivial extension of K by E. According to [18, Theorem 3.8], the ring $K(+)E$ is not $\{1\}(+)E$-Noetherian. Now, we will show that $K(+)E$ is a $\{1\}(+)E$-endo-Noetherian ring.
Let $(a, e) \in K(+)E\backslash\{(0, 0)\}$. Let $(b, f) \in ann_{K(+)E}(a, e)$. Then $(a, e)(b, f) = (ab, af + be) = (0, 0)$. Now if $a \neq 0$, then $b = 0$ and $f = 0$. If $a = 0$, then $e \neq 0$ and $be = 0$, so $b = 0$ and $f \in E$ is arbitrary. Hence

$$ann_{K(+)E}(a, e) = \begin{cases} (0, 0) & \text{if} \quad a \neq 0 \\ 0(+)E & \text{if} \quad a = 0 \end{cases} .$$

On the other hand, for every $g \in E$, we have

$$(1, g)ann_{K(+)E}(a, e) = \begin{cases} (0, 0) & \text{if} \quad a \neq 0 \\ 0(+)E & \text{if} \quad a = 0 \end{cases} .$$

Now, let $(a_k, e_k)_{k\geq 1}$ be a sequence of elements of $K(+)E$ such that

$$ann_{K(+)E}(a_1, e_1) \subseteq ann_{K(+)E}(a_2, e_2) \subseteq \cdots .$$

If $a_k = 0$ for each a positive integer $k \geq 1$, then $ann_{K(+)E}(a_k, e_k) = 0(+)E$ and for each $s \in \{1\}(+)E$, $s.ann_{K(+)E}(a_k, e_k) = ann_{K(+)E}(a_n, e_n)$ for each positive integer n. Let p be the first positive integer such that $a_p \neq 0$. Then for each positive integer $n \geq p$, we have $(1, g)ann_{K(+)E}(a_p, e_p) = 0(+)E = ann_{K(+)E}(a_p, e_p)$. Therefore the ring $K(+)E$ is a $\{1\}(+)E$-endo-Noetherian ring.

In the light of Example 2.2, the following proposition shows that, for a useful kind of condition, one can characterize when we have the converse.

Proposition 2.5. *Let A be a ring and S be a multiplicative subset of A such that $S \subseteq A \setminus Z(A)$. Then A is S-endo-Noetherian if and only if A is endo-Noetherian.*

Proof.
The "only if" half is trivial. Conversely, let $(a_k) \geq 1$ be a sequence of elements of A such that $ann(a_1) \subseteq ann(a_2) \subseteq \cdots$. Then, there exist $s \in S$ and $n \in \mathbb{N}$ such for each integer $k \geq n, \quad s.ann(a_k) \subseteq ann(a_n)$. Let $x \in ann(a_k)$. Then $sxa_n = 0$. As s is regular, we conclude that $xa_n = 0$. Hence $ann(a_k) \subseteq ann(a_n)$ for each positive integer $k \geq n$. Thus A is an endo-Noetherian ring. □

Proposition 2.6. *Let $A \subseteq B$ be an extension of commutative rings and $S \subseteq A$ be a multiplicative set. If B is an S-endo-Noetherian ring, then so is A.*

Proof.
Let $ann(a_1) \subseteq ann(a_2) \subseteq \cdots$ be an increasing sequence of annihilators of A, where $(a_k)_{k\geq 1}$ is a sequence of elements of A. Since B is S-endo-Noetherian, the sequence $(ann_B(a_k)_k)$ is S-stationary. So there exist $s \in S$ and a positive integer n such that for each $k \geq n, s(ann_B(a_k)) \subseteq ann_B(a_n)$. This implies that for each $k \geq n, s\,(ann_B(a_k) \cap A) \subseteq ann_B(a_n) \cap A$. Thus for each $k \geq n, s(ann_A(a_k)) \subseteq ann_A(a_n)$. Hence A is an S-endo-Noetherian ring. □

Let A be a ring and $S \subseteq A$ be a multiplicative set. We say that A satisfies the property $S-(*)$, if every increasing sequence of the form $(I : a_1) \subseteq (I : a_2) \subseteq \cdots$ is S-stationary, where I is an ideal of A and $(a_k)_k$ is a sequence of elements of A. It is clear that if A satisfies the property $S-(*)$, then A is S-endo-Noetherian. Note that if A satisfies the property $S-(*)$, then A satisfies the $S-accr$ condition (i.e, A is said to satisfy the $S-accr$ condition if the ascending chain of residuals of the form $(I : a) \subseteq (I : a^2) \subseteq \cdots$ is S-stationary for each $a \in A$ and every ideal I of A (cf. [13])).

Let A be a ring, S a multiplicative subset of A and I be an ideal of A. Set $\overline{S} := \{\overline{s}; s \in S\}$. It is clear that $\overline{S}$ is a multiplicative subset of A/I.

The following theorem characterizes the rings which satisfies the property $S-(*)$ in terms of S-endo-Noetherian rings.

Theorem 2.7. *Let A be a ring and $S \subseteq A$ a multiplicative set. The following assertions are equivalent:*

(1) A/I is an $\overline{S}$-endo-Noetherian ring for each ideal I of A such that $I \cap S = \emptyset$.
(2) A satisfies the property $S-(*)$.

Proof.
(1) ⇒ (2) Let I be an ideal of A such that $I \cap S = \emptyset$ and $(a_k)_k$ a sequence of elements of A with $(I : a_1) \subseteq (I : a_2) \subseteq \cdots$. We will show that $ann_{A/I}(\overline{a_k}) \subseteq ann_{A/I}(\overline{a_{k+1}})$. Let $\overline{x} \in ann_{A/I}(\overline{a_k})$. Then $xa_k \in I$, so $x \in (I : a_k) \subseteq (I : a_{k+1})$, which implies $xa_{k+1} \in I$. Therefore, $\overline{x} \in ann_{A/I}(a_{k+1})$. Now, since A/I is $\overline{S}$-endo-Noetherian, there exist $s \in S$ and a positive integer n such that for each $k \geq n$, $\overline{s}(ann_{A/I}(\overline{a_k})) \subseteq ann_{A/I}(\overline{a_n})$.

Let $x \in (I : a_k)$. Then $xa_k \in I$, which implies that $\overline{sxa_n} = \overline{0}$ in A/I. Hence $sxa_n \in I$, so $sx \in (I : a_n)$. Thus $s(I : a_k) \subseteq (I : a_n)$ for each $k \geq n$. Then A satisfies the property $S - (*)$.

(2) $\Rightarrow$ (1) Let $(a_k)_{k\geq 1}$ be a sequence of elements of A such that $ann_{A/I}(\overline{a_1}) \subseteq ann_{A/I}(\overline{a_2}\ ) \subseteq \cdots$. We show that $(I : a_1) \subseteq (I : a_2) \subseteq \cdots$. Let $x \in (I : a_k)$. Then $\overline{x} \in ann_{A/I}(\overline{a_k}) \subseteq ann_{A/I}(\overline{a_{k+1}})$. Thus $x \in (I : a_{k+1})$. Now since A satisfies the property $S-(*)$, there exist an $s \in S$ and a positive integer n such that for each $k \geq n$, $s(I : a_k) \subseteq (I : a_n)$. Let $\overline{x} \in ann_{A/I}(\overline{a_k})$. Then $xa_k \in I$, which implies $sx \in (I : a_n)$. This implies $\overline{sxa_n} = \overline{0}$ in A/I. Then $\overline{sx} \in ann_{A/I}(a_n)$, and hence $\overline{s}(ann_{A/I}(a_k)) \subseteq ann_{A/I}(\overline{a_n})$ for each integer $k \geq n$. Thus, A/I is an $\overline{S}$-endo-Noetherian ring. □

The previous theorem allows us to construct an example of a non S-Noetherian ring that satisfies the condition $S - (*)$.

Example 2.8. Let K be a field and E a K-vector space of infinite dimension. Consider $A := K(+)E$ to be the trivial ring extension of K by E. By [18, Theorem 3.8], the ring $A := K(+)E$ is not $\{1\}(+)E$- Noetherian. We will show that every ideal of $K(+)E$ is the form $0(+)F$, where F is a sub-space of E.

Indeed, let I be a proper ideal of $K(+)E$. Suppose that I contains an element (a, e) with $a \neq 0$. Then $(a, e).(\frac{1}{a}, -\frac{1}{a^2}e) = (1, 0) \in I$, hence $I = K(+)E$ since $(1, 0)$ is the unity of $K(+)E$. Thus each proper ideal I is contained in $0(+)E$. Using [1, Corollary 3.4], the proper ideal contained in $0(+)E$ is of the form $0(+)F$, where F is a sub-space of E. By [1, Theorem 3.1], we have $(K(+)E) \diagup (0(+)F) \simeq K(+)(E\diagup F)$.

Corollary 2.9. [13, Theorem 2.9] *The following assertions are equivalent for a ring A.*

(1) A/I is an endo-Noetherian ring for each ideal I of A.

(2) A satisfies the property $(*)$.

Proof. Follows from Theorem 2.7 with $S = \{1\}$. □

Proposition 2.10. *Let A, B be two rings and let $f : A \longrightarrow B$ be a ring homomorphism. Let S be a multiplicative subset of A. If R satisfies $S - (*)$, then $f(A)$ satisfies $f(S) - (*)$. In particular, $f(A)$ is an $f(S)$-endo-Noetherian ring.*

Proof.

Let I be an ideal of $f(R)$, and $a_1, a_2, \ldots \in R$ such that $(I :_{f(R)} f(a_1)) \subseteq (I :_{f(R)} f(a_2)) \subseteq \cdots$. We will show that $(f^{-1}(I) :_R a_1) \subseteq (f^{-1}(I) :_R a_2) \subseteq \cdots$. Let $x \in (f^{-1}(I) :_R a_i)$. Then $f(x) \in (I :_{f(R)} f(a_i))$. Therefore $f(x) \in (I :_{f(R)} f(a_{i+1}))$, and hence $x \in (f^{-1}(I) :_R a_{i+1})$.

Now, since R satisfies $S - (*)$, there exist an $s \in S$ and a positive integer n such that $s(f^{-1}(I) :_R a_k) \subseteq (f^{-1}(I) :_R a_n)$ for each $k \geq n$. We show that for each $k \geq n$, $f(s).(I :_{f(A)} a_k) \subseteq (I :_A a_n)$. Let $x \in A$ such that $f(x) \in$

$(I :_{f(A)} a_k)$, then $x \in (f^{-1}(I) :_A a_k)$ which implies $sx \in (f^{-1}(I) :_A a_n)$. Hence $f(s)f(x)f(a_n) \in I$. Therefore $f(s).(I :_{f(A)} a_k) \subseteq (I :_A a_n)$.

The "in particular" is clear. □

Theorem 2.11. *Let A and B be two rings and let S_1 and S_2 be a multiplicative subset of A and B, respectively: set $S := S_1 \times S_2$. Then the following statements are equivalents:*

(1) $A \times B$ is an S-endo-Noetherian ring.
(2) A is an S_1-endo-Noetherian ring and B is an S_2-endo-Noetherian ring.

Proof.
(1) ⇒ (2) Let $(a_k)_{k\in N^*}$ be a sequence of elements of A such that $ann_A(a_1) \subseteq ann_A(a_2) \subseteq \cdots$. It is easy to see that $ann_{A\times B}((a_1, 0)) \subseteq ann_{A\times B}((a_2, 0)) \subseteq \cdots$. Since $A\times B$ is an S-endo-Noetherian ring, there exist $(s, t) \in S = S_1\times S_2$ and a positive integer n such that $(s, t).ann((a_k, 0)) \subseteq ann((a_n, 0))$ for each $k \geq n$. Let $x \in ann(a_k)$ where $k \geq n$. Then $(s, t)(x, 0)(a_n, 0) = (0, 0)$, so $sxt \in ann(a_n)$. Hence $s.ann(a_k) \subseteq ann(a_n)$ for each $k \geq n$. Thus A is an S_1-endo-Noetherian ring. By the same argument we conclude that B is an S_2-endo-Noetherian ring.

(2) ⇒ (1) Let $(a_k, b_k)_{k\in\mathbb{N}^*}$ be a sequence of elements of $A\times B$ such that $ann_{A\times B}(a_1, b_1) \subseteq ann_{A\times B}(a_2, b_2) \subseteq \cdots$. It is easy to see that $ann_A(a_1) \subseteq ann_A(a_2) \subseteq \cdots$ and $ann_B(b_1) \subseteq ann_B(b_2) \subseteq \cdots$. Since A is an S_1-endo-Noetherian ring and B is an S_2-endo-Noetherian ring, there exist $s_1 \in S_1, s_2 \in S_2$ and a positive integer n such that $s_1.ann_A(a_k) \subseteq ann_A(a_n)$ and $s_2.ann_B(b_k) \subseteq ann_B(b_n)$. It is clear that $(s_1, s_2).ann_{A\times B}(a_k, b_k) \subseteq ann_{A\times B}(a_n, b_n)$ for each $k \geq n$. Hence $A \times B$ is an $S_1 \times S_2$-endo-Noetherian ring. □

Corollary 2.12. *Let $(A_i)_{1\leq i\leq n}$ be a finite family of rings and $A = \prod_{i=1}^n A_i$. For each $1 \leq i \leq n$ let S_i be a multiplicative subset of A_i. Set $S := \prod_{i=1}^n S_i$. Then, the following statements are equivalent:*

(1) A is an S-endo-Noetherian ring.
(2) A_i is an S_i-endo-Noetherian ring for each $1 \leq i \leq n$.

Proof.
Follows from Theorem 2.11. □

The following theorem characterizes when the duplication $A \bowtie I$ is an $S \times S$-endo-Noetherian ring.

Theorem 2.13. *Let A be a ring and S a multiplicative subset of A and I be a regular ideal of A. The following assertions are equivalent:*

(1) The ring A is S-endo-Noetherian.
(2) The duplication $A \bowtie I$ is an $S \times S$-endo-Noetherian ring.

Before proving Theorem 2.13, we first establish the following lemma.

Lemma 2.14. *Let A be a ring and I be a regular ideal of A. Let $(a, a+i), (b, b+j)$ two elements of $A \bowtie I$ such that $ann_{A\bowtie I}(a, a+i) \subseteq ann_{A\bowtie I}(b, b+j)$. Then $ann_{A\times A}(a, a+i) \subseteq ann_{A\times A}(b, b+j)$.*

Proof. Let $(x, y) \in A \times A$ such that $(x, y) \in ann_{A\times A}(a, a+i)$. Then $(x, y)(a, a+i) = (0, 0)$. Let k be a regular element of I. Then

$$(k, k)(x, y)(a, a+i) = (kx, kx + k(y-x))(a, a+i) = (0, 0).$$

Since $(kx, kx+k(y-x)) \in A \bowtie I$, we conclude by hypothesis that $(k, k)(x, y)(b, b+j) = (0, 0)$. Now, since (k, k) is a regular element of $A \times A$, we get $(x, y)(b, b+j) = (0, 0)$. Hence $ann_{A\times A}(a, a+i) \subseteq ann_{A\times A}(b, b+j)$. □

Proof of Theorem 2.13:

(1) ⇒ (2) Let $(a_k, a_k + i_k)_{k\in\mathbb{N}^*}$ be a sequence of elements of $A \bowtie I$ satisfying $ann_{A\bowtie I}(a_1, a_1 + i_1) \subseteq ann_{A\bowtie I}(a_2, a_2 + i_2) \subseteq \cdots$. By Lemma 2.14 we have $ann_{A\times A}(a_1, a_1 + i_1) \subseteq ann_{A\times A}(a_2, a_2 + i_2) \subseteq \cdots$. Using Theorem 2.11 there exist $(s, s) \in S \times S$ and a positive integer n such that for each $k \geq n, (s, s).ann_{A\times A}(a_k, a_k + i_k) \subseteq ann_{A\times A}(a_n, a_n + i_n)$, and then $(s, s).(A \bowtie I \cap ann_{A\times A}(a_k, a_k + i_k)) \subseteq A \bowtie I \cap ann_{A\times A}(a_n, a_n + i_n)$. So $(s, s).(ann_{A\bowtie I}(a_k, a_k+i_k)) \subseteq ann_{A\bowtie I}(a_n, a_n+i_n)$. Hence $A \bowtie I$ is an $S\times S$-endo-Noetherian ring.

(2) ⇒ (1) This implication is true even if I is not a regular ideal. Let $(a_k)_{k\in\mathbb{N}^*}$ be a sequence of elements of A satisfying $ann_A(a_1) \subseteq ann_A(a_2) \subseteq \cdots$. It is clear that $ann_{A\bowtie I}(a_1, a_1) \subseteq ann_{A\bowtie I}(a_2, a_2) \subseteq \cdots$. Since $A \bowtie I$ is an $S \times S$-endo-Noetherian ring, there exist $s \in S$ and a positive integer n such that for each $k \geq n, (s, s).(ann_{A\bowtie I}(a_k, a_k)) \subseteq ann_{A\bowtie I}(a_n, a_n)$. Now, let $x \in ann_A(a_k)$. Then we have $(x, x) \in ann_{A\bowtie I}(a_k, a_k)$, which implies $(s, s)(x, x)(a_n, a_n) = (0, 0)$. Hence $sx \in ann_A(a_n)$. Finally A is an S-endo-Noetherian ring. □

Corollary 2.15. [12, Theorem 2.13] *Let A be a ring and I be a regular ideal of A. The following assertions are equivalent:*

(1) The ring A is an endo-Noetherian.
(2) The product ring $A \times A$ is endo-Noetherian ring.
(3) The duplication $A \bowtie I$ is an endo-Noetherian ring.

Proof.

(1) ⟺ (2) Follows from Theorem 2.11 with $S = \{1\}$.
(1) ⟺ (3) Follows from Theorem 2.13 with $S = \{1\}$. □

3. Polynomials and Formal power series over a S-endo-Noetherian ring

Recall from [16] that a commutative ring is called an Armendariz ring if whenever the polynomials $f = \sum_{i=0}^{n} a_i X^i$ and $g = \sum_{i=0}^{m} b_i X^i \in R[X]$ with $f.g = 0$, then $a_i b_j = 0$ for every i and j.

Remark 3.1. Let R be an Armendariz ring. Let $f = \sum_{i=0}^{n} a_i X^i$ and set $K = \{a_i | 0 \leq i \leq n\}$. The fact that R is an Armendariz ring gives that $\text{ann}_R(K)[X] = \text{ann}_{R[X]}(f)$. In particular, if K is a finite subset of R and $f \in R|X]$ a polynomial whose set of coefficients is K, then $\text{ann}_R(K)[X] = \text{ann}_{R[X]}(f)$.

Remark 3.2. If $I = (a_0, ..., a_n)$ is a finitely generated ideal of a ring R, then $\text{ann}_R(I) = \text{ann}_R(K)$, where $K = \{a_i | 0 \leq i \leq n\}$.

Theorem 3.3. *Let R be an Armendariz ring and S be a multiplicative subset of R. Then the following statements are equivalent:*

(1) $R[X]$ is an S-endo-Noetherian ring.
(2) R satisfies the $S - acc$ on annihilators of finite subsets.
(3) R satisfies the $S - acc$ on annihilators of finitely generated ideals of R.

Proof.

(1) $\Rightarrow$ (2) Let $(K_i)_{i \in \mathbb{N}^*}$ be a sequence of non-empty finite subsets of R such that $\text{ann}_R(K_1) \subseteq \text{ann}_R(K_2) \subseteq \cdots$. Clearly we have $\text{ann}_R(K_1)[X] \subseteq \text{ann}_R(K_2)[X] \subseteq \cdots$. We consider $f_i(X) \in R[X]$, a polynomial whose set of coefficients of f_i is K_i for each $i \in \mathbb{N}^*$. By Remark 3.1, we have $\text{ann}_{R[X]}(f_1) \subseteq \text{ann}_{R[X]}(f_2) \subseteq \cdots$. Now, since $R[X]$ is an S-endo-Noetherian ring, there exist $s \in S$ and a positive integer n such that $s(\text{ann}_{R[X]}(f_k)) \subseteq \text{ann}_{R[X]}(f_n)$ for every positive integer $k \geq n$, which implies $s(\text{ann}(K_k)[X]) \subseteq \text{ann}(K_n)[X]$. Hence, we conclude that there exist $s \in S$ and a positive integer n such that $s(\text{ann}(K_k)) \subseteq \text{ann}(K_n)$ for every $k \geq n$.

(2) $\Rightarrow$ (3) Follows from (2) and Remark 3.2.

(3) $\Rightarrow$ (1) Let $(f_i)_{i \in \mathbb{N}^*}$ be a sequence of elements of $R[X]$ satisfying $\text{ann}_{R[X]}(f_1) \subseteq \text{ann}_{R[X]}(f_2) \subseteq \cdots$. For each $i \in \mathbb{N}^*$, let K_i be the set of the coefficients of f_i. Since R is an Armendariz ring, we conclude by Remark 3.1, that $(\text{ann}_{R[X]}(K_1))[X] \subseteq (\text{ann}_{R[X]}(K_2))[X] \subseteq \cdots$. Then $\text{ann}_R(K_1) \subseteq \text{ann}_R(K_2) \subseteq \cdots$. Now, for each $i \in \mathbb{N}^*$, let I_i be the ideal generated by K_i. According to Remark 3.2, we have $\text{ann}_R(I_1) \subseteq \text{ann}_R(I_2) \subseteq \cdots$. Since (3) holds, there exist $s \in S$ and a positive integer n such that $s(\text{ann}_R(I_k) \subseteq \text{ann}_R(I_n)$, and so $s(\text{ann}_R(K_k) \subseteq \text{ann}_R(K_n)$ for every positive integer $k \geq n$. Now, the result follows from Remark 3.1. This completes the proof. $\square$

Lemma 3.4. *Let R be a power series Armendariz ring, $f(X) \in R[[X]]$, and L the set of the coefficients of $f(X)$. Then, $(ann_R(L))\,[[X]] = ann_{R[[X]]}(f)$.*

Proof.

Let $g(X) = \sum_{i=0}^{\infty} b_i X^i \in R[[X]]$. Then $g(X) \in (\text{ann}_R(L))\,[[X]]$, if and only if for every $i \in \mathbb{N}$, $b_i \in \text{ann}_R(L)$ if and only if for every $i \in \mathbb{N}$, $ab_i = 0$ for an arbitrary element a in L, that is $g(X)f(X) = 0$, which is equivalent to saying that $g(X) \in \text{ann}_{R[[X]]}(f)$. This completes the proof of the lemma. $\square$

Theorem 3.5. *Let R be an Armendariz ring and S be a multiplicative subset of R. Then the following statements are equivalent:*

(1) $R[[X]]$ is an S-endo-Noetherian ring.
(2) The ring R satisfies $S-acc$ for the annihilators of the countably sets.
(3) The ring R satisfies $S-acc$ for the annihilators of the countably generated ideals.

Proof.
(1) $\Rightarrow$ (2) Let $(L_i)_{i\in\mathbb{N}^*}$ be a sequence of countably subsets of A such that $\text{ann}_R(L_1) \subseteq \text{ann}_R(L_2) \subseteq \cdots$. Then $\text{ann}_R(L_1)[[X]] \subseteq \text{ann}_R(L_2)[|X]] \subseteq \cdots$. For each $i \in \mathbb{N}^*$, we consider $f_i(X) \in R[[X]]$ whose set of coefficients is equal to L_i. By Lemma 3.4, we have $\text{ann}_{R[[X]]}(f_1(X)) \subseteq \text{ann}_{R[[X]]}(f_2(X)) \subseteq \cdots$. Now, since $R[[X]]$ is an S-endo-Noetherian ring, there exist $n \in \mathbb{N}^*$ and $s \in S$ such that $s.\text{ann}_{R[[X]]}(f_k(X)) \subseteq \text{ann}_{R[[X]]}(f_n(X))$ for each $k \geq n$. This implies that $s.\text{ann}_R(L_k)[[X]] \subseteq \text{ann}_R(L_n)[[X]]$, therefore $s.\text{ann}_R(L_k) \subseteq \text{ann}_R(L_n)$ for each $n \geq n$, as desired.

(2) $\Rightarrow$ (1) Assume that (2) holds and let $(f_i)_{i\in\mathbb{N}^*}$ be a sequence of elements of $R[[X]]$ satisfying $\text{ann}_{R[[X]]}(f_1) \subseteq \text{ann}_{R[[X]}(f_2) \subseteq \cdots$. For each $i \in \mathbb{N}^*$, we denote by L_i the set of the coefficients of f_i. By Lemma 3.4 we have $\text{ann}_R(L_1)[[X]] \subseteq \text{ann}_R(L_2)[[X]] \subseteq \cdots$, which implies that $\text{ann}_R(L_1) \subseteq \text{ann}_R(L_2) \subseteq \cdots$. By assumption there exist $n \in \mathbb{N}^*$ and $s \in S$ such that $s.\text{ann}_R(L_k) \subseteq \text{ann}_R(L_n)$. Now the result follows from Lemma 3.4.

(2) $\Rightarrow$ (3) Follows from the fact that if I is a countably generated ideal of R whose countable set formed by the elements that generates I is S, then $\text{ann}_R(I) = \text{ann}_R(S)$. □

Corollary 3.6. *Let R be an Armendariz ring and $S \subseteq R$ be a multiplicative set. Then the following statements are equivalent:*

(1) $R[[X]]$ is an S-endo-Noetherian ring.
(2) For each sequence $(f_k)_{k\in\mathbb{N}^*}$ of elements of $R[[X]]$, $f_k = \sum_{j\geq 0} a_{k,j}X^j$ satisfying $ann_{R[[X]]}(f_1) \subseteq ann_{R[[X]]}(f_2) \subseteq \cdots$, then there exist $n \in \mathbb{N}^*$ and $s \in S$ such that for each $k \geq n$, we have $s.\bigcap_{j\geq 0} ann_R(a_{k,j}) \subseteq \bigcap_{j\geq 0} ann_R(a_{n,j})$.

Proof.
Let $f = \sum_{i\geq 0} a_i X^i$ and let L be the set of coefficients of f. It is enough to note that $\text{ann}_R(K) = \bigcap_{i\geq 0} \text{ann}_R(a_i)$. Now, the result follows immediately from (1) $\Leftrightarrow$ (2) of Theorem 3.5. □

Acknowledgement

The authors would like to thank the referee for his/her great efforts in proofreading the manuscript.

References

1. D. D. Anderson and T. Dumitrescu, *S-Notherian rings*, Comm. Algebra, 30(9) (2002), 4407–4419.
2. D. D. Anderson and M. Winders, *Idealization of a module*, J. Commut. Algebra, 1 (1) (2009), 3–56.
3. J. Baeck, G. Lee and J. W. Lim, *S-Noetherian rings and their extension*, Taiwaness J. Math, 20 (6), (2016), 1231–1250.
4. A. Benhissi, Chain Conditions in Commutative Rings, Springer Nature (2022).
5. A. Benhissi, *Chain condition on annihilators and strongly Hopfian property in Hurwitz series ring*, Algebra Colloq., 21 (4) (2014), 635–646.
6. M. B. Boisen and P. B. Sheldon, *CPI-extension: Over rings of integral domains with special prime spectrum*, Canad. J. Math, 29 (4) (1977), 722–737.
7. M. D'Anna, C. A. Finocchiaro and M. Fontana, *Amalgamated algebras along an ideal*, Comm. Algebra and Applications, Walter de Gruyter, Berlin, (2009), 155–172.
8. M. D'Anna, *A construction of Gorenstein rings*, J. Algebra, 306 (2) (2006), 507–519.
9. M. D'ana and M. Fontana, *The amalgamated duplication of a ring along a multiplication-canonical ideal*, Ark. Mat., 45 (2) (2007), 241–252.
10. M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl., 6 (3) (2007), 443–459.
11. A. El Khalfi, H. Kim and N. Mahdou, *Amalgamation extension in commutative ring theory, a survey*, Moroccan Journal of Algebra and Geometry with Applications, 1 (1) (2022), 139–182.
12. B. Gouaid, A. Hamed and A. Benhissi, *Endo-Noetherian rings*, Ann. Mat. Pur. Appl., 199 (2) (2020), 563–572.
13. A. Hamed and S. Hizem, *Modules Satisfying the S-Noetherian Property and S-ACCR*, Comm. Algebra, 44 (5) (2006), 1941–1951.
14. H. Hamed and S Hizem, *S-Noetherian rings of the form A[X] and A[[X]]*, Comm. Algebra, 43 (9) (2015), 3848–3856.
15. A. Hamed and S. Hizem, *Modules satisfying the S-Noetherian property and S-ACCR*, Comm. Algebra, 44 (5) (2016), 1941–1951.
16. N. K. Kim and Y. Lee, *Armendariz ring and reduced ring*, J. Algbra, 223 (2) (2000), 477–488.
17. J. W. Lim, *A note on S-Noetherian domains*, Kyungpook Math. J., 55 (3) (2015), 507–514.
18. J. W. Lim and D.Y. Oh, *S-Noetherian properties on amalgamated algebras along an ideal*, J. Pure Appl. Algebra, 218 (6) (2014), 1075–1080.
19. M. Nagata, Local Rings, Wiley-Interscience, New York, 1962.
20. M. A. Ndiaye and C. T. Gueye, *On commutative EKFN-ring with ascending chain condition on annihilators*, Int. J. App. Math., 86 (5) (2013), 871–881.
21. E. Sengelen Sevim, U. Tekir and S. Koc, *S-Artinian rings and finitely S-cogenerated rings*, J. Algebra Appl., 19 (3) (2020), 2050051.

Mohammed Issoual, Departement of Mathematics CRMEF RABAT-SALÉ-KENITRA: Annexe CRMEF KHMISSET, Morocco.

E − mail address : *issoual2@yahoo.fr*

Salah Eddine Mahdou, Laboratory of Modelling and Mathematical Structures,, Faculty of Science and Technology of Fez, Box 2202, University S. M. Ben Abdellah Fez, Morocco.

E − mail address : *salahmahdoulmtiri@gmail.com*